Sample complexity trade-offs for synthetic aperture based high-resolution estimation and detection

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***Abstract*—This paper critically examines the potential performance benefits offered by motion of sparse arrays for direction-of-arrival (DOA) estimation. The motivation behind utilizing array motion is to increase the number of consecutive difference lags. However, creating a synthetic array also requires more temporal measurements compared to the static (non-synthetic) array. For the first time, we rigorously analyze the trade-off between the required number of temporal samples and the length of the difference co-array to understand when synthetic arrays offer distinct advantages. As a concrete result, we show that if the ratio of the number of consecutive lags of the difference coarray of sparse arrays with and without motion is above a universal threshold, the synthetic array outperforms its non-synthetic counterpart and has a smaller estimation error. Our claims are demonstrated both theoretically and through numerical experiments[1].**

***Index Terms* — Sparse Array, Difference Co-Array, Array Motion, Synthetic Array, Sample Complexity Trade-Off.**

I. Introduction

The problem of direction-of-arrival (DOA) estimation involves inferring the directions of electromagnetic sources received at an antenna array [5], [2]. If the sensor array is a uniform linear array (ULA) of size M, it is well-known that the number (Q) of uniquely identifiable sources satisfies the bound $Q < M$. In a series of work [6], [7], [8], it has been established that the restriction $Q < M$ can be overcome when the incoming sources are statistically uncorrelated. The main idea in [6], [7] is to design suitable non-uniform sparse arrays that can exploit this statistical property and create the effect of a virtual difference co-array with an enlarged aperture of size $O(M^2)$. Algorithms such as Co-array MUSIC [9] and Co-array LASSO [8] can recover more sources than sensors by utilizing the difference co-array of these sparse arrays.

In [10], [3], the authors consider the scenario when the sparse array is mounted on a moving platform. By introducing array motion, it is possible to obtain a synthetic array after combining the original sparse array and its shifted version. The primary benefit of the synthetic array is its ability to fill holes in the difference co-array of the static version. This increases the length of the contiguous segment in the difference array, and subsequently increase the number of sources that can be identified. In [3], the authors present a detailed discussion on this advantage for common sparse array designs such as nested, coprime, minimum redundancy array (MRA) and minimum hole array (MHA). It is also empirically shown that array motion can indeed improve the performance of DOA estimation by enlarging the virtual array aperture. However, the idea of constructing a synthetic array as proposed in [3] requires more temporal measurements than the static non-synthetic array. Since the statistical performance of DOA estimation is also dependent on the number of snapshots [4], [11] in addition to the virtual array aperture, it is of great importance to analyze the trade-off between the temporal sample complexity and the expansion in size of difference co-array due to motion. Such statistical performance analysis has not been provided in existing literature.

The contributions of this paper are two fold. Firstly, we propose a unified analysis of DOA estimation error for both synthetic and non-synthetic arrays by using the analysis techniques developed in [4]. In particular, we develop statistical guarantees on the estimation error that explicitly characterize the number of required temporal measurements. Secondly, we compare the trade-off between the virtual array aperture and temporal sample measurements based on these derived error bounds. Several concrete sparse array configurations are discussed to demonstrate when non-synthetic arrays may be preferable. One of our main conclusions is to realize that the key quantity of interest is the ratio of the lengths of the ULA segments in the difference co-arrays of the synthetic and non-synthetic configurations, which also controls the number of required temporal measurements. If this ratio is above a certain threshold, the synthetic array is recommended since it will provably offer performance benefits.

Throughout the paper, we use $\odot$ and $*$ to denote the Khatri-Rao product and complex conjugate respectively. The cardinality of a set $\mathbb{S}$ is denoted by $|\mathbb{S}|$.

II. Signal Model based on Array Motion

Consider a sparse array with M_s antennas on a platform moving at a constant speed v. The array receives narrowband signals $s_q(t)$, $q = 1, 2, \cdots, Q$ around a carrier of wavelength λ from far-field sources whose directions of arrival (DOA) are given by θ_q. Assuming that the output of the array is sampled at a frequency of $1/T_s$ Hz, the measurement vector $\mathbf{x}(lT_s) \in \mathbb{C}^{M_s}$ at time $t = lT_s$ is given by:

$$\mathbf{x}(lT_s) = \sum_{q=1}^{Q} s_q(lT_s) e^{-j\frac{2\pi v l T_s \sin(\theta_q)}{\lambda}} \mathbf{a}(\theta_q) + \mathbf{n}(lT_s) \qquad (1)$$

Here $\mathbf{n}(t)$ denotes additive noise and

$$\mathbf{a}(\theta) = \left[1, e^{-j\pi d_1 \sin(\theta)}, \cdots, e^{-j\pi d_{M_s-1} \sin(\theta)}\right]^T$$

is the array steering vector at time $t = 0$ with d_m being the location of the m^{th} sensor normalized with respect to $\lambda/2$ (assuming the first sensor to be at 0). Throughout this paper, we make the following statistical assumptions:

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[A1]: Let $\mathbf{s}[l] = [s_1(lT_s), s_2(lT_s), \cdots, s_Q(lT_s)]^T$. We assume that $\mathbf{s}[l]$ are zero-mean i.i.d random vectors with $\mathbb{E}\left(\mathbf{s}[l]\mathbf{s}[l]^H\right) = \text{diag}(\mathbf{p})$ where $\mathbf{p} = [p_1, p_2, \cdots, p_Q]^T$ denotes the source powers.

[A2]: The noise $\mathbf{n}[l]$ follows a zero-mean complex Gaussian distibution and is uncorrelated with $\mathbf{s}[l]$, satisfying $\mathbb{E}\left(\mathbf{n}[l]\mathbf{n}[l]^H\right) = \sigma^2\mathbf{I}_{M_s}$. The noise power σ^2 is assumed to be known.

The classical measurement model without array motion ($v = 0$) is given by [5], [6], [7]

$$\mathbf{x}[l] = \mathbf{x}(lT_s) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}[l] + \mathbf{n}[l], \quad l = 1, \cdots, L \tag{2}$$

where $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \cdots \mathbf{a}(\theta_Q)] \in \mathbb{C}^{M_s \times Q}$. Under assumption [A1], the covariance matrix of $\mathbf{x}[l]$ is given by:

$$\mathbf{R}_{\mathbf{xx}} = \mathbb{E}\mathbf{x}[l]\mathbf{x}[l]^H = \mathbf{A}\text{diag}(\mathbf{p})\mathbf{A}^H + \sigma^2\mathbf{I}_{M_s} \tag{3}$$

which can be reformulated as (3) as

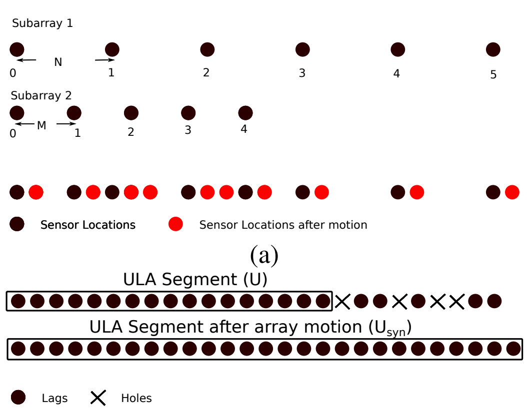
$$\mathbf{r}_{\mathbf{x}} = \text{vec}(\mathbf{R}_{\mathbf{xx}}) = (\mathbf{A}^* \odot \mathbf{A})\mathbf{p} + \sigma^2\text{vec}(\mathbf{I}_{M_s}) \tag{4}$$

As shown in [6], [7], [8], [11], [4], the matrix $\mathbf{A}^* \odot \mathbf{A}$ can be identified as the array manifold of a virtual sensor array, the location of whose elements are given by the following set:

$$\mathbb{D} := \{d_m - d_n, 1 \leqslant m, n \leqslant M_s\} \tag{5}$$

The set $\mathbb{D}$ is also called as the "difference set" of the set of physical antenna locations. For a well-designed sparse array, the size of $\mathbb{D}$ can be as large as $|\mathbb{D}| = O(M_s^2)$.

In [3], the authors exploit the motion of sparse arrays to create a synthetic array whose difference set can be even larger. In Figure 1, we show an example of the static and synthetic array with motion. Since the source signal $s_q(t)$ are



(b)

Fig. 1: (a) Sensor locations for a coprime array with $2M + N - 1$ sensors ($M = 3$ and $N = 5$) before and after motion. (b) Non-negative half of the difference co-array of the Non-synthetic (Top) and Synthetic array (Bottom)

narrowband with respect to a carrier frequency f, for small enough τ, we have $s_q(lT_s + \tau) \approx s_q(lT_s)e^{j2\pi f\tau}$ [3]. Denoting $d_s := v\tau/(\lambda/2)$ (τ is chosen such that d_s is an integer[2]), the shifted array steering vector is defined as [3]

$$\mathbf{b}(\theta_q) = e^{-j\frac{2\pi d_s \sin(\theta_q)}{\lambda}}\mathbf{a}(\theta_q) \tag{6}$$

[2]In [3], the authors choose $d_s = 1$

And the measurements at $lT_s + \tau$ are given by

$$\mathbf{x}(lT_s + \tau) = e^{j2\pi f\tau}\mathbf{B}\mathbf{D}^l\mathbf{s}[l] + \mathbf{n}(lT_s + \tau)$$

where $\mathbf{B} = [\mathbf{b}(\theta_1), \cdots, \mathbf{b}(\theta_Q)] \in \mathbb{C}^{M_s \times Q}$ and

$$\mathbf{D} = \text{diag}(e^{-j\frac{2\pi v T_s \sin(\theta_1)}{\lambda}}, \cdots, e^{-j\frac{2\pi v T_s \sin(\theta_Q)}{\lambda}}) \in \mathbb{C}^{Q \times Q} \tag{7}$$

represents the phase-shift of the steering vector because of the array motion. After compensating for the additional (known) phase $e^{j2\pi f\tau}$, we obtain [12]

$$\tilde{\mathbf{x}}(lT_s + \tau) = e^{-j2\pi f\tau}\mathbf{x}(lT_s + \tau) = \mathbf{B}\mathbf{D}^l\mathbf{s}[l] + \tilde{\mathbf{n}}(lT_s + \tau)$$

Combining $\mathbf{x}[l]$ and $\tilde{\mathbf{x}}(lT_s + \tau)$, we obtain the output of the synthetic array as

$$\begin{aligned}\mathbf{y}[l] &= \begin{bmatrix} \mathbf{x}(lT_s) \\ \tilde{\mathbf{x}}(lT_s + \tau) \end{bmatrix} \\ &= \mathbf{A}_{\text{syn}}(\boldsymbol{\theta})\mathbf{D}^l\mathbf{s}[l] + \begin{bmatrix} \mathbf{n}(lT_s) \\ \tilde{\mathbf{n}}(lT_s + \tau) \end{bmatrix}\end{aligned} \tag{8}$$

where $\mathbf{A}_{\text{syn}}(\boldsymbol{\theta}) = [\mathbf{a}_{\text{syn}}(\theta_1), \cdots, \mathbf{a}_{\text{syn}}(\theta_Q)] \in \mathbb{C}^{2M_s \times Q}$ and $\mathbf{a}_{\text{syn}}(\theta) = [\mathbf{a}(\theta)^T, \mathbf{b}(\theta)^T]^T$ denotes the overall steering vector of the synthetic array. The difference co-array of the synthetic array, denoted $\mathbb{D}_{\text{syn}}$, is generated by considering the difference set of sensors at both their original locations and after being shifted by d_s units due to array motion. It is given by

$$\mathbb{D}_{\text{syn}} = \mathbb{D} \bigcup \{d_m - d_n \pm d_s, 1 \leqslant m, n, \leqslant M_s\} \tag{9}$$

where $\mathbb{D}$ denotes the difference co-array of the non-synthetic array, defined in (5). Evidently, $|\mathbb{D}_{\text{syn}}| \geqslant |\mathbb{D}|$. Let $\mathbb{U}_{\text{syn}}$ and $\mathbb{U}$ denote the maximum ULA segment around 0 in $\mathbb{D}_{\text{syn}}$ and $\mathbb{D}$ respectively. Consider the covariance matrix of the synthetic array, $\mathbf{R}_{\mathbf{yy}} := E\left(\mathbf{y}[l]\mathbf{y}^H[l]\right)$. By retaining only those entries of the matrix $\mathbf{R}_{\mathbf{yy}} - \sigma^2\mathbf{I}_{2M_s}$ which are indexed by $\mathbb{U}_{\text{syn}}$, we obtain the vector

$$\mathbf{r}_{\mathbf{y}}^{\mathbb{U}_{\text{syn}}} = \mathbf{V}_{\mathbb{U}_{\text{syn}}}\mathbf{p}_0 \tag{10}$$

Here, $\mathbf{V}_{\mathbb{U}_{\text{syn}}} \in \mathbb{C}^{|\mathbb{U}_{\text{syn}}| \times Q}$ consists of rows of the *synthetic difference co-array* manifold $\mathbf{A}_{\text{syn}}^* \odot \mathbf{A}_{\text{syn}}$ which are indexed by $\mathbb{U}_{\text{syn}}$. The vector $\mathbf{p}_0 \in \mathbb{R}^Q$ consists of the diagonal entries of the matrix $\mathbf{D}^l\mathbb{E}\left(\mathbf{s}[l]\mathbf{s}[l]^H\right)\left(\mathbf{D}^l\right)^H$. Although the synthetic array has a larger difference co-array and hence offers extra spatial degrees of freedom, we pay an additional price in terms of the need to collect additional (in fact, twice) temporal measurements. It is therefore important to understand under what conditions synthetic array, equipped with a larger difference set, provably leads to a smaller estimation error with finite temporal snapshots.

III. Spatio-temporal Sample Complexity Trade-off in Synthetic Arrays

In this section, we analyze the performance of sparse synthetic arrays with motion by explicitly characterizing the trade-offs between the size of the difference co-array $|\mathbb{D}_{\text{syn}}|$ and temporal measurements L. We build on key ideas from our recent paper on non-asymptotic performance analysis of sparse arrays [4]. Denoting $\omega := \pi\sin(\theta)$ as the spatial frequency, we discretize the set of spatial angles into K grid points given by $\omega_k = -\pi + 2\pi k/K, k = 0, 1, \cdots K - 1$. Using this grid based model, we can re-write (10) as [8], [3], [4]

$$\mathbf{r}_{\mathbf{y}}^{\mathbb{U}_{\text{syn}}} = \tilde{\mathbf{V}}_{\mathbb{U}_{\text{syn}}}\tilde{\mathbf{p}}_0 \tag{11}$$

where $\tilde{\mathbf{V}}_{\mathbb{U}_{\text{syn}}} \in \mathbb{C}^{|\mathbb{U}_{\text{syn}}| \times K}$ and $\tilde{\mathbf{p}}_0 \in \mathbb{C}^K$ denote the quantities as in (10) only defined on a grid of size K. The vector $\tilde{\mathbf{p}}_0 \in \mathbb{C}^K$ is sparse with $Q << K$ non-zero entries. The indices of the non-zero elements of $\tilde{\mathbf{p}}_0$ reveal the desired DOAs on the grid. Hence, by recovering $\tilde{\mathbf{p}}_0$, it is possible to estimate the DOAs. In practice, we estimate $\mathbf{R}_{\mathbf{yy}}$ using a finite number (L) of snapshots as $\hat{\mathbf{R}}_{\mathbf{yy}} = \frac{1}{L}\sum_l \mathbf{y}[l]\mathbf{y}[l]^H$. We can rewrite (11) using this estimate as

$$\hat{\mathbf{r}}_{\mathbf{y}}^{\mathbb{U}_{\text{syn}}} = \tilde{\mathbf{V}}_{\mathbb{U}_{\text{syn}}}\tilde{\mathbf{p}}_0 + \mathbf{\Delta}_L \tag{12}$$

where $\hat{\mathbf{r}}_{\mathbf{y}}^{\mathbb{U}_{\text{syn}}}$ is the finite snapshot estimate of $\mathbf{r}_{\mathbf{y}}^{\mathbb{U}_{\text{syn}}}$, obtained from the sample covariance matrix. The vector $\mathbf{\Delta}_L$ depends on L and represents the finite snapshot estimation error. The statistical properties of $\mathbf{\Delta}_L$ plays crucial role in determining the performance of synthetic arrays with limited snapshots.

Our goal is to estimate the sparse vector $\tilde{\mathbf{p}}_0$ since the non-zero entries of this vector serve as estimates of the DOAs. Following [8], [3], we propose to recover $\tilde{\mathbf{p}}_0$ by solving the following optimization problem for synthetic array:

$$\min_{\mathbf{z}} \|\mathbf{z}\|_1 \quad \text{s.t. } \|\hat{\mathbf{r}}_{\mathbf{y}}^{\mathbb{U}_{\text{syn}}} - \tilde{\mathbf{V}}_{\mathbb{U}_{\text{syn}}}\mathbf{z}\|_2 \leqslant \epsilon,\ \mathbf{z} \geq \mathbf{0} \quad (P_{\text{Co-den,syn}})$$

Similarly, let $\tilde{\mathbf{V}}_{\mathbb{U}} \in \mathbb{C}^{|\mathbb{U}| \times K}$ represent the discretized array manifold corresponding to the ULA segment $\mathbb{U}$ in the difference co-array of the non-synthetic array. We solve the following problem to recover the sparse vector $\tilde{\mathbf{p}}_0$ of source powers

$$\min_{\mathbf{z}} \|\mathbf{z}\|_1 \quad \text{s.t. } \|\hat{\mathbf{r}}_{\mathbf{y}}^{\mathbb{U}} - \tilde{\mathbf{V}}_{\mathbb{U}}\mathbf{z}\|_2 \leqslant \epsilon,\ \mathbf{z} \geq \mathbf{0} \quad (P_{\text{Co-den}})$$

Leveraging recent results from [4], we will analyze and compare the performance of both $(P_{\text{Co-den}})$ and $(P_{\text{Co-den, syn}})$. We first introduce the notion of a separation condition that will be imposed on the support of $\tilde{\mathbf{p}}_0$.

Definition 1. *(Set of Non-negative Signals Obeying Separation Condition) Given K and $\mathbb{U}$, define the set $\mathcal{P}_{sep}^{+}$*

$$\mathcal{P}_{sep}^{+} \triangleq$$
$$\{\mathbf{p} \in \mathbb{C}^K \mid \mathbf{p} \geq \mathbf{0}, \phi(\frac{k}{K}, \frac{l}{K}) \geqslant \frac{4}{|\mathbb{U}|-1}, \forall k \neq l \in \mathit{Supp}(\mathbf{p})\}$$

where $\phi(\cdot,\cdot): \mathbb{R}^2 \to \mathbb{R}_+$ is a wrap-around distance function.

We utilize the following main result from [4].

Theorem 1. *Let $\{\mathbf{y}[l]\}_{l=0}^{L-1}$ be i.i.d zero-mean circularly-symmetric complex Gaussian random vectors with covariance matrix $\mathbf{R}_{\mathbf{yy}}$. Suppose $|\mathbb{U}| \geqslant 257$, $K \geqslant 3.03|\mathbb{U}|$ and $\rho = 0.0295\left(\frac{|\mathbb{U}|-1}{2K}\right)^2$. If the sparse vector $\tilde{\mathbf{p}}_0$ obeys $\tilde{\mathbf{p}}_0 \in \mathcal{P}_{sep}^{+}$, then for any $\epsilon > 0$ and $\delta \in (0,1)$ the solution $\mathbf{p}^{\#}$ to $(P_{\text{Co-den}})$ satisfies*

$$\|\mathbf{p}^{\#} - \tilde{\mathbf{p}}_0\|_1 \leqslant 2\epsilon(\frac{1-\rho}{\rho}) \tag{13}$$

with a probability at least $1-\delta$ provided

$$L \geqslant \max\left\{\frac{2\text{trace}^2(\mathbf{R}_{\mathbf{yy}})}{\epsilon^2}, \left(\frac{\log\frac{2}{\delta}}{2c}\right)^2\right\} \tag{14}$$

where c is a positive constant.

To apply Theorem 1 to the synthetic array, we need to enforce that the collected snapshots are i.i.d complex Gaussian. The following lemma establishes that indeed, $\{\mathbf{D}^l\mathbf{s}[l]\}_{l=0}^{L-1}$ continue to be i.i.d zero-mean circularly-symmetric Gaussian random vectors.

Lemma 1. *If $\{\mathbf{s}[l]\}_{l=0}^{L-1}$ are i.i.d circularly-symmetric complex Gaussian random vectors with zero-mean independent coordinates, then the snapshots $\{\mathbf{D}^l\mathbf{s}[l]\}_{l=0}^{L-1}$ are also independent vectors with the same distribution.*

Before we state our main result, we introduce some important quantities below which will determine the overall sample complexity:

$$\rho_{\text{syn}} = 0.0295(\frac{|\mathbb{U}_{\text{syn}}|}{2K})^2, \rho = 0.0295(\frac{|\mathbb{U}|}{2K})^2$$
$$C_p = 2M^2\frac{(\|\tilde{\mathbf{p}}_0\|_1 + \sigma^2)^2}{\epsilon^2}$$

We characterize the total number of temporal measurements L_T required by both synthetic and non-synthetic array configurations. For a synthetic array, a total of L_T measurements corresponds to $L_T/2$ snapshots acquired at both the original and shifted positions. Now, we are ready the state the main result about the performance of the synthetic array specified by (8).

Theorem 2. *Suppose $\{\mathbf{s}[l]\}_{l=0}^{L-1}$ are i.i.d circularly-symmetric complex Gaussian vectors. If $\tilde{\mathbf{p}}_0 \in \mathcal{P}_{sep}^{+}$, $|\mathbb{U}_{syn}| \geqslant 257$ and $K \geqslant 3.03|\mathbb{U}_{syn}|$, then for a given $\delta \in (0,1)$, the solution $\mathbf{p}^{\#}$ of $(P_{\text{Co-den,syn}})$ with any $\epsilon > 0$ satisfies*

$$\|\mathbf{p}^{\#} - \tilde{\mathbf{p}}_0\|_1 \leqslant 2\epsilon$$

with probability at least $1-\delta$ if

$$L \geqslant \max\left\{\frac{4C_p}{\rho_{syn}^2}, \left(\frac{\log\frac{2}{\delta}}{2c}\right)^2\right\}.$$

Proof. Follows directly from Theorem 1 by substituting ϵ as $\frac{\rho_s\epsilon}{1-\rho_s}$ □

Similarly, we can obtain the following result for non-synthetic array where we solve $(P_{\text{Co-den}})$.

Corollary 1. *Suppose $\{\mathbf{s}[l]\}_{l=0}^{L-1}$ are i.i.d circularly-symmetric complex Gaussian vectors. If $\tilde{\mathbf{p}}_0 \in \mathcal{P}_{sep}^{+}$, $|\mathbb{U}| \geqslant 257$ and $K \geqslant 3.03|\mathbb{U}|$, the solution $\mathbf{p}^{\#}$ of (P_{Co-den}) with any $\epsilon > 0$ satisfies*

$$\|\mathbf{p}^{\#} - \tilde{\mathbf{p}}_0\|_1 \leqslant 2\epsilon$$

with probability at least $1-\delta$ if

$$L \geqslant \max\left\{\frac{C_p}{\rho^2}, \left(\frac{\log\frac{2}{\delta}}{2c}\right)^2\right\}.$$

A. Comparison of Sample Complexities for Synthetic and Non-Synthetic Arrays

We now compare the sample complexities of the two array configurations to develop an understanding of the regime where synthetic array offers performance benefits over the non-synthetic array. Specifically, suppose we want both arrays to achieve a desired upper bound of 2ϵ on the estimation error with the same probability of $1-\delta$. Since we use L snapshots to estimate the covariance matrix for both arrays, Theorem 2 dictates that the total number of temporal measurements for the synthetic array should satisfy $L_T = 2L \geqslant \max\left\{\frac{8C_p}{\rho_{\text{syn}}}, 2\left(\frac{\log\frac{2}{\delta}}{2c}\right)^2\right\}$. A simple computation reveals that if

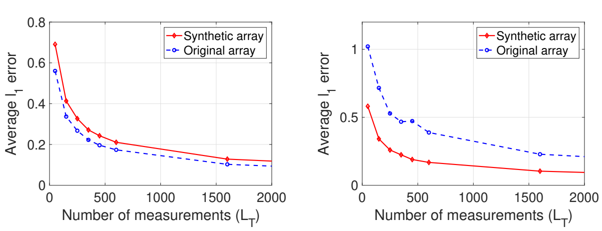
$$\frac{\rho_{\text{syn}}}{\rho} > 2\sqrt{2}$$

the synthetic array can achieve the same bound on estimation error with fewer temporal measurements. The ratio ρ_{syn}/ρ is determined by the length of the central ULA segment of the synthetic and non-synthetic co-arrays. As shown in the following table, even with the same number of sensors M_s, this quantity can have significantly different behavior depending on the array geometry.

Array	Number of sensors	$\lvert\mathbb{U}\rvert$ (Non-Syn)	$\lvert\mathbb{U}_{\text{syn}}\rvert$ (Syn)	$\frac{\rho_{\text{syn}}}{\rho}$
Conf 1 $M=3, N=5$	10	35	53	2.34
Conf 2 $M=5, N=6$	10	21	59	4.41
Conf 1 $M=4, N=5$	12	47	65	1.94
Conf 2 $M=4, N=9$	12	25	59	5.8403

TABLE I. Effect of Array Geometry on the ratio $\frac{\rho_{\text{syn}}}{\rho}$

Example: We consider two different coprime array configurations to illustrate the dependence of the ratio ρ_{syn}/ρ on the specific array geometry. Configuration I refers to the coprime array from [7] consisting of two subarrays with $2M$ and N sensors (M and N being coprime integers) and respective inter-sensor spacings given by N and M. Configuration II refers to the coprime array studied in [3], where the subarrays consist of M and N elements, with N and M being the respective intersensor spacings. In Table 1, we can see that for the same number of sensors, the ratio ρ_{syn}/ρ exceeds the threshold $2\sqrt{2}$ only with Configuration II. This is because Configuration II has holes in the co-array, which get filled due to array motion, resulting in higher performance gains.

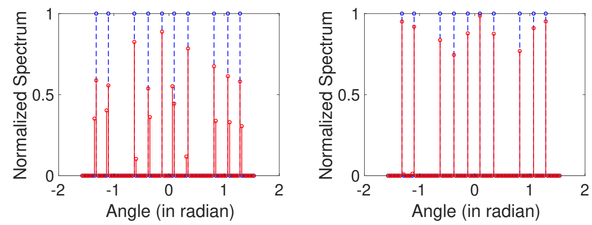


(a) (b)

Fig. 2: Estimation error v/s L_T for different Coprime Array configurations with 10 sensors (a) Coprime array in configuration I with $M=3$ and $N=5$ (b) Coprime array in configuration II with $M=5$ and $N=6$

IV. Simulations

In our first experiment, we compare the estimation error from solving ($P_{\text{Co-den}}$) and ($P_{\text{Co-den,syn}}$) for the non-synthetic and synthetic arrays respectively, using coprime arrays in two different configurations. We consider 10 sensors and $Q=17$ sources for configuration I, and $Q=8$ sources for configuration II. The sources are placed randomly on a grid of size 100. We evaluate the performance of both configurations by computing the l_1 norm of the estimation error $\frac{\|\mathbf{p}^{\#}-\tilde{\mathbf{p}}_0\|_1}{\|\tilde{\mathbf{p}}_0\|_1}$ averaged over 100 Monte Carlo trials, as the number of measurements (L) varies from 50 to 6000. The SNR is chosen to be 0dB for both configurations. Figure 2 (a) demonstrates that for configuration I, the non-synthetic array requires fewer measurements than synthetic array in order to achieve a desired estimation error, as predicted by our theoretical analysis. In Figure 2 (b), we observe that the synthetic array outperforms non-synthetic array in configuration II. This observation is consistent with the theoretical result as the ratio $\rho_{\text{syn}}/\rho > 2\sqrt{2}$ for this array configuration.



(a) (b)

Fig. 3: Sparse Support Recovery with $Q=10$ sources and 10 sensors using a coprime array in Configuration II. (a) True support (blue) and its estimate (red) using non-synthetic coprime array. (b) True support (blue) and its estimate (red) using synthetic coprime array.

In the next experiment, we study the support recovery performance of synthetic and non-synthetic coprime arrays in configuration II. We consider 10 sensors and a total of $L_T = 2000$ measurements for both synthetic and non-synthetic arrays. For this experiment, we choose $Q=10$ sources which are placed non-uniformly over the interval $[-\pi,\pi]$. In Figure 3, the synthetic array (b) outperforms the fixed array (a). This happens because both configurations have different separation condition determined by their ULA segment. The synthetic array has a less stringent separation requirement due to a larger ULA segment. This demonstrates that synthetic arrays can allow higher resolution DOA estimates compared to non-synthetic arrays.

V. Conclusion

In this paper, we developed statistical guarantees for DOA estimation with sparse arrays in motion, which explicitly reveals the trade-off between spatial and temporal measurements. Our results demonstrate the potential benefits offered by array motion under appropriate conditions. In particular, we showed that if the ratio of the number of consecutive difference lags with and without array motion is larger than a specific threshold, synthetic arrays can achieve the same error as static arrays with potentially fewer temporal snapshots. Our analysis can be used in practice to make an informed choice between synthetic and non-synthetic configurations.

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